

On the nonexistence of $\left[\binom{2m}{m-1}, 2m, \binom{2m-1}{m-1}\right]$, m odd, complex orthogonal design

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Abstract

Complex orthogonal designs (CODs) are used to construct space-time block codes. COD \mathcal{O}_z with parameter $[p, n, k]$ is a $p \times n$ matrix, where nonzero entries are filled by $\pm z_i$ or $\pm z_i^*$, $i = 1, 2, \dots, k$, such that $\mathcal{O}_z^H \mathcal{O}_z = (|z_1|^2 + |z_2|^2 + \dots + |z_k|^2) I_{n \times n}$. Adams et al. in “The final case of the decoding delay problem for maximum rate complex orthogonal designs,” IEEE Trans. Inf. Theory, vol. 56, no. 1, pp. 103-122, Jan. 2010, first proved the nonexistence of $\left[\binom{2m}{m-1}, 2m, \binom{2m-1}{m-1}\right]$, m odd, COD. Combining with the previous result that decoding delay should be an integer multiple of $\binom{2m}{m-1}$, they solved the final case $n \equiv 2 \pmod{4}$ of the decoding delay problem for maximum rate complex orthogonal designs.

In this paper, we give another proof of the nonexistence of COD with parameter $\left[\binom{2m}{m-1}, 2m, \binom{2m-1}{m-1}\right]$, m odd. Our new proof is based on the uniqueness of $\left[\binom{2m}{m-1}, 2m-1, \binom{2m-1}{m-1}\right]$ under equivalence operation, where an explicit-form representation is proposed to help the proof. Then, by proving it's impossible to add an extra orthogonal column on COD $\left[\binom{2m}{m-1}, 2m-1, \binom{2m-1}{m-1}\right]$ when m is odd, we complete the proof of the nonexistence of COD $\left[\binom{2m}{m-1}, 2m, \binom{2m-1}{m-1}\right]$.

Key words: complex orthogonal design, space-time block codes, maximal rate and minimal delay.

1 Introduction

Space-time block codes have been widely investigated for wireless communication systems with multiple transmit and receive antennas. Since the pioneering work by Alamouti [5] in 1998, and the work by Tarokh et al. [13], [14], orthogonal designs have become an effective technique for the design of space-time block codes (STBC). The importance of this class of codes comes from the fact that they achieve full diversity and have the fast maximum-likelihood (ML) decoding.

A complex orthogonal design (COD) $\mathcal{O}_z[p, n, k]$ is an $p \times n$ matrix, and each entry is filled by $\pm z_i$ or $\pm z_i^*$, $i = 1, 2, \dots, k$, such that $\mathcal{O}_z^H \mathcal{O}_z = \sum_{i=1}^n |z_i|^2 I_n$, where H is the Hermitian transpose and I_n is the $n \times n$ identity matrix. Under this definition, the designs are said to be combinatorial, in the sense that there is no linear processing in each entry. Code rate k/p and decoding delay p are the two most important criteria of complex orthogonal space-time block codes.

One important problem is, given n , determine the tight upper bound of code rate, which is called maximal rate problem. Another is, given n , determine the tight lower bound of decoding delay p when code rate k/p reaches the maximal, which is called minimal delay problem.

For combinatorial CODs, where linear combination is not allowed, Liang determined for a COD with $n = 2m$ or $2m - 1$, the maximal possible rate is $\frac{m+1}{2m}$ [9]. Liang gave an algorithm in [9] to generate such CODs with rate $\frac{m+1}{2m}$, which shows that this bound is tight. The minimal delay problem are solved by Adams et al. In [3], lower bound $\binom{2m}{m-1}$ of decoding delay is proved for any $n = 2m$ or $2m - 1$. And further, it's proved that the decoding delay must be a multiple of $\binom{2m}{m-1}$. In [4], by showing the nonexistence of COD $\left[\binom{2m}{m-1}, 2m, \binom{2m-1}{m-1}\right]$ with even m , Adams et al. prove that when $n \equiv 2 \pmod{4}$, decoding delay p is lowered bound by $2\binom{2m}{m-1}$.

The tightness of above upper bound of rate is shown by constructions in [9], [12], [11]. When $n \not\equiv 0 \pmod{4}$, those in [9], [12] also achieve the lower bound of delay. And constructions in [11] achieves minimal delay for all n .

The organization of our paper is as follows. In section 2, we introduce the notions, definitions and some known results which will be used. In section 3, we present our main results including the uniqueness of COD with parameter $\left[\binom{2m}{m-1}, 2m - 1, \binom{2m-1}{m-1}\right]$, and the nonexistence of COD having parameter $\left[\binom{2m}{m-1}, 2m, \binom{2m-1}{m-1}\right]$ which depends on the former result. In order to prove the main results, an explicit-form construction of optimal COD is introduced, which is crucial to our proofs.

2 Preliminaries

In this section, we introduce some basic notions, which will be used in the sequel.

\mathbb{C} denotes the field of complex numbers, \mathbb{R} the field of real numbers and \mathbb{F}_2 the field with two elements. Adding over \mathbb{F}_2 is denoted by \oplus to avoid ambiguity. All vectors are assumed to be column vectors. For any field \mathbb{F} , denoted by \mathbb{F}^n and $M_{m \times n}(\mathbb{F})$ the set of all n -dimensional vectors in \mathbb{F} and the set of all $m \times n$ matrices in \mathbb{F} , respectively. In this paper, rows and variables are often indexed by vectors in \mathbb{F}_n^2 .

For convenience, let $e_i \in \mathbb{F}_2^n$ be the vector with i^{th} bit occupied by 1 and the others 0, i.e., $e_i = (\underbrace{0, \dots, 0}_{i-1}, 1, \underbrace{0, \dots, 0}_{n-i})$ and let $e = e_1 \oplus e_2 \oplus \dots \oplus e_n$, i.e.,

$e = (1, 1, \dots, 1)_2$. The weight of a vector in \mathbb{F}_2^n is defined as the number of ones in n bits, i.e., $\text{wt}(\alpha) = \sum_{i=1}^n \alpha(i)$. Furthermore, $\text{wt}_{s,t}(\alpha)$ is defined as the sum of s^{th} bit to t^{th} bit, i.e.,

$$\text{wt}_{s,t}(\alpha) = \alpha(s) + \alpha(s+1) + \dots + \alpha(t) = \sum_{i=s}^t \alpha(i).$$

In abuse of notation, we denote by $z[j]$ the complex variable z_j , up to negation and conjugation, i.e., $z[j] \in \{z_j, -z_j, z_j^*, -z_j^*\}$. Note that the same notation $z[j]$ may represent different elements in the same paragraph.

Definition 2.1. A $[p, n, k]$ complex orthogonal design \mathcal{O}_z is a $p \times n$ rectangular matrix whose nonzero entries are $z_1, z_2, \dots, z_k, -z_1, -z_2, \dots, -z_k$ or their conjugates $z_1^*, z_2^*, \dots, z_k^*, -z_1^*, -z_2^*, \dots, -z_k^*$, where z_1, z_2, \dots, z_k are indeterminates over \mathbb{C} , such that

$$\mathcal{O}_z^H \mathcal{O}_z = (|z_1|^2 + |z_2|^2 + \dots + |z_k|^2) I_{n \times n}.$$

k/p is called the code rate of \mathcal{O}_z , and p is called the decoding delay of \mathcal{O}_z .

A matrix is called an Alamouti 2×2 if it matches the following form

$$\begin{pmatrix} z_i & z_j \\ -z_j^* & z_i^* \end{pmatrix}, \quad (1)$$

up to negation or conjugation of z_i or z_j . We say two rows share an Alamouti 2×2 if and only if the intersection of the two rows and some two columns form an Alamouti 2×2 .

Definition 2.2. The equivalence operations performed on any COD are defined as follows.

- 1) Rearrange the order the rows (“row permutation”).
- 2) Rearrange the order the columns (“column permutation”).
- 3) Conjugate all instances of certain variable (“instance conjugation”).
- 4) Negate all instances of certain variable (“instance negation”).
- 5) Change the index of all instances of certain variable (“instance renaming”).
- 6) Multiply any row by -1 , (“row negation”).
- 7) Multiply any column by -1 , (“column negation”).

It's not difficult to verify that, given a COD $\mathcal{O}_z[p, n, k]$, after arbitrary equivalence operations, we will obtain another COD $\mathcal{O}'_z[p, n, k]$. And we say COD \mathcal{O}_z and \mathcal{O}'_z are the same under equivalence operations.

Following the definition in [9], with a little modification, define an (n_1, n_2) - \mathcal{B}_j form by

$$\begin{aligned} \mathcal{B}_j &= \begin{pmatrix} z_j I_{n_1} & \mathcal{M}_1 \\ -\mathcal{M}_1^H & z_j^* I_{n_2} \end{pmatrix} \\ &= \left(\begin{array}{cccc|cccc} z_j & 0 & \cdots & 0 & & & & \\ 0 & z_j & \cdots & 0 & & & & \\ \vdots & \vdots & \ddots & \vdots & & & & \\ 0 & 0 & \cdots & z_j & & & & \\ \hline & & & & -\mathcal{M}_j^H & & & \\ & & & & & z_j^* & 0 & \cdots & 0 \\ & & & & & 0 & z_j^* & \cdots & 0 \\ & & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & & 0 & 0 & \cdots & z_j^* \end{array} \right), \quad (2) \end{aligned}$$

where $n_1 + n_2 = n$. And we call it \mathcal{B}_j form for short.

Definition 2.3. [3] We say COD \mathcal{O}_z is in \mathcal{B}_j form if the submatrix \mathcal{B}_j can be created from \mathcal{O}_z through equivalence operations except for column permutation. Equivalently, \mathcal{O}_z is in \mathcal{B}_j form if every row of \mathcal{B}_j appears within the rows of \mathcal{O}_z , up to possible conjugations of all instances of z_i and possible factors of -1 .

It is proved that [3] that COD \mathcal{O}_z is in some \mathcal{B}_j form if and only if one row in \mathcal{O}_z matches one row of \mathcal{B}_j up to signs and conjugations.

In [9], Liang proved the upper bound $\frac{m+1}{2m}$ of code rate $\frac{k}{p}$ for any $n = 2m$ or $2m - 1$, and obtained the necessary and sufficient condition to reach the maximal rate.

Theorem 2.4. Let $n = 2m$ or $2m - 1$. The rate of COD $\mathcal{O}_z[p, n, k]$ is upper bounded by $\frac{m+1}{2m}$, i.e., $\frac{k}{p} \leq \frac{m+1}{2m}$.

This bound is achieved if and only if for all $i = 1, 2, \dots, k$, \mathcal{B}_j is an $(m, m - 1)$ - \mathcal{B}_j or $(m - 1, m)$ - \mathcal{B}_j form and there are no zero entries in \mathcal{M}_j , when $n = 2m - 1$; \mathcal{B}_j is an (m, m) - \mathcal{B}_j form and there are no zero entries in \mathcal{M}_j , when $n = 2m$.

The lower bound on the decoding delay when code rate reaches the maximal is completely solved by Adams et al. in [3] and [4].

Theorem 2.5. Let $n = 2m$ or $2m - 1$. For COD $\mathcal{O}_z[p, n, k]$, if the rate reaches the maximal, i.e., $\frac{k}{p} = \frac{m+1}{2m}$, the delay, i.e., p , is lower bounded by $\binom{2m}{m-1}$ when $n \equiv 0, 1, 3 \pmod{4}$; by $2\binom{2m}{m-1}$ when $n \equiv 2 \pmod{4}$.

The technique in proving the lower bound $\binom{2m}{m-1}$ is the observation and definition of zero pattern, which is a vector in \mathbb{F}_2^n defined with respect to one row where the i^{th} bit is 0 if and only if the element on column i is 0. For example, when

$$\mathcal{O}_z = \begin{pmatrix} z_1 & z_2 & z_3 \\ -z_2^* & z_1^* & 0 \\ -z_3^* & 0 & z_1^* \\ 0 & z_3^* & -z_2^* \end{pmatrix}, \quad (3)$$

the first row has zero pattern $(1, 1, 1)$, the second $(1, 1, 0)$, the third $(1, 0, 1)$, the fourth $(0, 1, 1)$.

In [3], it's proved that the decoding delay is an integer multiple of $\binom{2m}{m-1}$. Therefore, in order to prove the lower bound of delay $2\binom{2m}{m-1}$ for $n \equiv 2 \pmod{4}$, it's sufficient to prove the nonexistence of $\left[\binom{2m}{m-1}, 2m, \binom{2m-1}{2m-1}\right]$. The basic idea in [4] is, first proving the uniqueness of COD with parameter $\left[\binom{2m}{m-1}, 2m - 1, \binom{2m-1}{2m-1}\right]$ under equivalence operation, then showing is impossible to add an extra column for a specific COD with parameter $\left[\binom{2m}{m-1}, 2m - 1, \binom{2m-1}{2m-1}\right]$ to obtain a new one. Our proof follows the same basic idea, but different from theirs, we define an explicit-form COD $\mathcal{G}_{2m-1} \left[\binom{2m}{m-1}, 2m - 1, \binom{2m-1}{2m-1}\right]$, while another standard form is defined to help prove the uniqueness in [4]. Due to our explicit-form construction, it's much easier to show the impossibility of adding an extra orthogonal column to \mathcal{G}_{2m-1} .

3 Main Results

The following lemma is first proved in [4], which an observation of \mathcal{B}_j form.

Lemma 3.1. *For maximal rate COD $\mathcal{O}_z[p, n, k]$, if $|\mathcal{O}_z(\alpha, i)| = |\mathcal{O}_z(\beta, j)|$ then the zero patterns of row α and row β are only different at column i and j ; if $|\mathcal{O}_z(\alpha, i)| = |\mathcal{O}_z(\beta, j)|^*$ then the zero pattern of row α and row β are only the same at column i and j .*

The following lemma is also first proved in [3]. For completeness, we give another proof.

Lemma 3.2. *For maximal rate COD $\mathcal{O}_z[p, n, k]$, $n = 2m$ or $2m - 1$, then $p \geq \binom{2m}{m-1}$. When $n = 2m$, every zero pattern with $m + 1$ ones exist; when $n = 2m - 1$, every zero pattern with m or $m + 1$ ones exist.*

Proof. First, we will prove if one zero pattern of some row is $\alpha \in \mathbb{F}_2^n$, then for any $\alpha(i) = 1, \alpha(j) = 0$, there exists one row with zero pattern $\beta \in \mathbb{F}_2^n$, such that $\beta(i) = \alpha(j), \beta(j) = \alpha(i)$ and $\beta(l) = \alpha(l)$ for all $l \neq i, j$. To see the existence of zero pattern β , we only need to arrange \mathcal{O}_z into \mathcal{B}_γ form, where $\mathcal{O}_z(\alpha, i) = z[\gamma]$.

Then, since any permutation is a product of transpositions, all zero patterns with weight m (or $m + 1$) exists. \square

As a consequence of Lemma 3.1 and Lemma 3.2, we know, up to negations, CODs with parameter $\left[\binom{2m}{m-1}, 2m - 1, \binom{2m-1}{m-1}\right]$ are the same under equivalence operation. The following lemma is to help define the “standard form” COD $\mathcal{G}_{2m-1} \left[\binom{2m}{m-1}, 2m - 1, \binom{2m-1}{m-1}\right]$.

Lemma 3.3. [2] *Let \mathcal{O}_z be a maximum rate, minimal delay COD with parameter $\left[\binom{2m}{m-1}, 2m - 1, \binom{2m-1}{m-1}\right]$. Then \mathcal{O}_z is equivalent to a COD that is conjugation separated, where the rows containing m nonzero entries are all conjugated, and those containing $m + 1$ nonzero entries are all non-conjugated.*

By Lemma 3.3, we know \mathcal{O}_z can be made conjugation separated with rows containing m nonzero entries all conjugated. We identify each row by its zero pattern and with the $2m^{\text{th}}$ bit denoting whether the row is conjugated or not, i.e., $\alpha \in \mathbb{F}_2^{2m}$ with $\text{wt}(\alpha) = m + 1$.

Let \mathcal{G}_{2m-1} be a COD with parameter $\left[\binom{2m}{m-1}, 2m - 1, \binom{2m-1}{m-1}\right]$ with rows identified by vectors in \mathbb{F}_2^{2m} with weight $m + 1$ and columns identified by $1, 2, \dots, 2m - 1$. The elements of \mathcal{G}_{2m-1} are determined by the following rules.

- If $\alpha(i) = 1$ and $\alpha(2m) = 0$, $\mathcal{O}_z(\alpha, i) = (-1)^{\theta(\alpha, i)} z_{\alpha \oplus e_i}$;
- If $\alpha(i) = 1$ and $\alpha(2m) = 1$, $\mathcal{O}_z(\alpha, i) = (-1)^{\theta(\alpha, i)} z_{\alpha \oplus e_i \oplus e}^*$;
- If $\alpha(i) = 0$, $\mathcal{O}_z(\alpha, i) = 0$.

Here

$$\theta(\alpha, i) = \begin{cases} \text{wt}_{i, 2m}(\alpha) + \frac{i}{2}, & \text{if } i \text{ is even,} \\ \text{wt}_{i, 2m}(\alpha) + \frac{i-1}{2} + \alpha(2m), & \text{if } i \text{ is odd.} \end{cases} \quad (4)$$

From Lemma 3.1, we know rows $\alpha, \beta \in \mathbb{F}_2^{2m}$ share an Alamouti 2×2 if and only if their zero patterns are only the same at column $1 \leq i, j \leq 2m - 1$, such

that $\alpha(i) = \alpha(j) = \beta(i) = \beta(j) = 1$ and $\alpha \oplus \beta = e \oplus e_i \oplus e_j$. Submatrix $\mathcal{G}_{2m-1}(\alpha, \beta; i, j)$ has the following form

$$\begin{pmatrix} (-1)^{\theta(\alpha, i)} z_{\alpha \oplus e_i} & (-1)^{\theta(\alpha, j)} z_{\alpha \oplus e_j} \\ (-1)^{\theta(\beta, i)} z_{\beta \oplus e_i \oplus e}^* & (-1)^{\theta(\beta, j)} z_{\beta \oplus e_j \oplus e}^* \end{pmatrix}$$

or

$$\begin{pmatrix} (-1)^{\theta(\alpha, i)} z_{\alpha \oplus e_i}^* & (-1)^{\theta(\alpha, j)} z_{\alpha \oplus e_j}^* \\ (-1)^{\theta(\beta, i)} z_{\beta \oplus e_i \oplus e} & (-1)^{\theta(\beta, j)} z_{\beta \oplus e_j \oplus e} \end{pmatrix}.$$

Note that $\alpha \oplus \beta = e \oplus e_i \oplus e_j \Rightarrow \alpha \oplus e_i = \beta \oplus e_j \oplus e$. Thus, we only need to check the signs to see whether submatrix $\mathcal{G}_{2m-1}(\alpha, \beta; i, j)$ is an Alamouti 2×2 .

Let's calculate $\theta(\alpha, i) + \theta(\beta, i)$ according to the parity of i , by definition (4). When i is even, $\theta(\alpha, i) + \theta(\beta, i) = \text{wt}_{i, 2m}(\alpha) + \frac{i}{2} + \text{wt}_{i, 2m}(\beta) + \frac{i}{2} \equiv \text{wt}_{i, 2m}(\alpha \oplus \beta) + i \pmod{2}$; When i is odd, $\theta(\alpha, i) + \theta(\beta, i) = \text{wt}_{i, 2m}(\alpha) + \frac{i-1}{2} + \alpha(2m) + \text{wt}_{i, 2m}(\beta) + \frac{i-1}{2} + \beta(2m) \equiv \text{wt}_{i, 2m}(\alpha \oplus \beta) + i \pmod{2}$. Therefore, we see

$$\theta(\alpha, i) + \theta(\beta, i) \equiv \text{wt}_{i, 2m}(\alpha \oplus \beta) + i \pmod{2} \quad (5)$$

always holds.

Then,

$$\begin{aligned} & \theta(\alpha, i) + \theta(\beta, i) + \theta(\alpha, j) + \theta(\beta, j) \\ \equiv & \text{wt}_{i, 2m}(\alpha \oplus \beta) + i + \text{wt}_{j, 2m}(\alpha \oplus \beta) + j \\ \equiv & \text{wt}_{i, j}(\alpha \oplus \beta) + i + j \\ \equiv & j - i - 1 + i + j \\ \equiv & 1 \pmod{2}. \end{aligned}$$

In the last second step, $\text{wt}_{i, j}(\alpha \oplus \beta) = j - i - 1$ is true because $\alpha \oplus \beta = e \oplus e_i \oplus e_j$.

Up to now, by constructing a specific function $\theta(\alpha, i)$, we see \mathcal{G}_{2m-1} is a COD. In fact, we only need to know such arrangement of signs exists.

Theorem 3.4. *Let \mathcal{O}_z be a COD with parameter $\left[\binom{2m}{m-1}, 2m-1, \binom{2m-1}{m-1} \right]$. Then \mathcal{O}_z is equivalent to \mathcal{G}_{2m-1} under equivalence operation.*

Proof. The basic idea is to show \mathcal{O}_z and \mathcal{G}_{2m-1} can be transformed into a standard form COD. Since equivalence operations are invertible, we claim \mathcal{O}_z is equivalent to \mathcal{G}_{2m-1} .

Before defining standard form, we first introduce a total order on vectors of \mathbb{F}_2^{2m} , that $\alpha < \beta \Leftrightarrow \sum_{i=1}^{2m} \alpha(i)2^i < \sum_{i=1}^{2m} \beta(i)2^i$. Now, we will show \mathcal{O}_z , as well as \mathcal{G}_{2m-1} , can be transformed into a “standard form” uniquely. Consider variable $z[\gamma]$ by increasing order of γ , where $\gamma \in \mathbb{F}_2^{2m}$, $\text{wt}(\gamma) = 2m$ and $\gamma(2m) = 0$. Keep in mind that our algorithm determines the signs of all instance of $z[\gamma]$ once at a time, and once the signs are determined, it will never change in

subsequent steps. Take out all rows containing $z[\gamma]$, which is

$$\begin{aligned} \mathcal{B}_\gamma &= \begin{pmatrix} \pm z_\gamma I_{n_1} & \mathcal{M}_\gamma \\ \mathcal{M}'_\gamma & \pm z_\gamma^* I_{n_2} \end{pmatrix} \\ &= \left(\begin{array}{cccc|cccc} \pm z_\gamma & 0 & \cdots & 0 & & & & \\ 0 & \pm z_\gamma & \cdots & 0 & & & & \\ \vdots & \vdots & \ddots & \vdots & & & & \\ 0 & 0 & \cdots & \pm z_\gamma & & & & \\ \hline & & & & \mathcal{M}'_\gamma & & & \\ \hline & & & & & \pm z_\gamma^* & 0 & \cdots & 0 \\ & & & & & 0 & \pm z_\gamma^* & \cdots & 0 \\ & & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & & 0 & 0 & \cdots & \pm z_\gamma^* \end{array} \right), \quad (6) \end{aligned}$$

where $n_1, n_2 \in \{m, m-1\}$ and $n_1 + n_2 = 2m - 1$.

Followings are two steps of our algorithm

- For those $z[\gamma]$ whose index are not smallest on the corresponding row, we will show there are only two possible ways to determine their signs. In other words, their relationships, same or opposite, are fixed due to the determined signs of $z[\delta]$, where $\delta < \gamma$. At last, we make use of instance negation to make sure, on the smallest row, $z[\gamma]$ is positive.
- If, in one row, $z[\gamma]$ is the element with the smallest index, which implies all other elements in the same row are undetermined, we can use row negation to make sure it's positive without affecting the determined signs.

Now, we prove the claim that “there are only two possible ways to determine their signs” in the first step above is true. Let $\gamma(1) = \gamma(2) = \dots = \gamma(s) = 1, \gamma(s+1) = 0, \gamma(t-1) = 1, \gamma(t) = \gamma(t+1) = \dots = \gamma(2m) = 0$, where $0 \leq s < t \leq 2m$. And assume $\mathcal{O}_z(\alpha_i, i) = z[\gamma]$ for $1 \leq i \leq 2m-1$, which implies $\alpha_i = \gamma \oplus e_i$ when $\gamma(i) = 0$, $\alpha_i = \gamma \oplus e_i \oplus e$ when $\gamma(i) = 1$.

For any $1 \leq i \neq j \leq 2m-1$, consider the element in $\mathcal{O}_z(\alpha_i, j)$. When $\alpha_i(j) = 0 \Leftrightarrow \gamma(i) = \gamma(j)$, $\mathcal{O}_z(\alpha_i, j) = 0$ by definition. When $\alpha_i(j) = 1 \Leftrightarrow \gamma(i) \oplus \gamma(j) = 1$, $\mathcal{O}_z(\alpha_i, j) = z[\delta]$, where $\delta = \alpha_i \oplus e_j \oplus \alpha_i(2m)e = \gamma \oplus e_i \oplus \gamma(i)e \oplus e_j \oplus \gamma(i)e = \gamma \oplus e_i \oplus e_j$. Therefore, for $1 \leq i \leq s$ or $t \leq i \leq 2m-1$, $\mathcal{O}_z(\alpha_i, i)$ is the element with the smallest index on that row.

For $s < i < t$ and $1 \leq j \leq 2m-1$ satisfying $\gamma(i) \oplus \gamma(j) = 1$, submatrix

$$\begin{pmatrix} \mathcal{O}_z(\alpha_i, i) & \mathcal{O}_z(\alpha_i, j) \\ \mathcal{O}_z(\alpha_j, i) & \mathcal{O}_z(\alpha_j, j) \end{pmatrix} = \begin{pmatrix} z[\gamma] & z[\gamma \oplus e_i \oplus e_j] \\ z[\gamma \oplus e_i \oplus e_j] & z[\gamma] \end{pmatrix}$$

is an Alamouti 2×2 . As our algorithm determines the signs of $z[\gamma]$ by increasing order, the signs of $z[\gamma \oplus e_i \oplus e_j]$ s are determined if and only if $\gamma \oplus e_i \oplus e_j < \gamma \Leftrightarrow \gamma(i) = 1, \gamma(j) = 0, i > j$ or $\gamma(i) = 0, \gamma(j) = 1, i < j$. Therefore, if $\gamma(i) = 0$, the relationship of signs of $\mathcal{O}_z(\alpha_i, i)$ and $\mathcal{O}_z(\alpha_{t-1}, t-1)$ are determined; if $\gamma(i) = 1$, the relationship of signs of $\mathcal{O}_z(\alpha_i, i)$ and $\mathcal{O}_z(\alpha_{s+1}, s+1)$ are determined. Since the relationship of signs of $\mathcal{O}_z(\alpha_{s+1}, s+1)$ and $\mathcal{O}_z(\alpha_{t-1}, t-1)$ are determined, and by the transitivity of sign relationship, we claim all $\mathcal{O}_z(\alpha_i, i)$ for $s < i < t$ are uniquely determined. \square

It's worth noting that in the proof of Theorem 3.4, "column negation" operation is not used. Therefore, any COD with parameter $\left[\binom{2m}{m-1}, 2m-1, \binom{2m-1}{m-1}\right]$ can be transformed into \mathcal{G}_{2m-1} without using "column negation" operation.

Lemma 3.5. *When m is odd, it's impossible to obtain a COD with parameter $\left[\binom{2m}{m-1}, 2m, \binom{2m-1}{m-1}\right]$ by adding an extra column on \mathcal{G}_{2m-1} .*

Proof. Assume that there exists such a $\left[\binom{2m}{m-1}, 2m, \binom{2m-1}{m-1}\right]$ COD by adding an extra column on \mathcal{G}_{2m-1} . Denote the last column by \mathcal{L}_{2m} , and assume $\mathcal{O}_z = (\mathcal{G}_{2m-1}, \mathcal{L}_{2m})$ is a $\left[\binom{2m}{m-1}, 2m, \binom{2m-1}{m-1}\right]$ COD.

By Lemma 3.1, we know \mathcal{L}_{2m} , up to negations, are uniquely determined. It's not difficult to verify that $\mathcal{L}_{2m}(\alpha) = \alpha(2m)(-1)^{\phi(\alpha)} z_{\alpha \oplus e_{2m}}$ for $\alpha \in \mathbb{F}_2^{2m}$ and $\text{wt}(\alpha) = m+1$, where $\phi(\alpha) \in \mathbb{F}_2$ are undetermined.

For any $\alpha \in \mathbb{F}_2^{2m}$ with $\text{wt}(\alpha) = m+1$ and $\alpha(2m) = 1$, $\mathcal{O}_z(\alpha, i)$ and $\mathcal{O}_z(\alpha, 2m)$ are contained in the following Alamouti 2×2

$$\begin{pmatrix} \mathcal{O}_z(\alpha, i) & \mathcal{O}_z(\alpha, 2m) \\ \mathcal{O}_z(\beta, i) & \mathcal{O}_z(\beta, 2m) \end{pmatrix} = \begin{pmatrix} (-1)^{\theta(\alpha, i)} z_{\alpha \oplus e_i \oplus e}^* & (-1)^{\phi(\alpha)} z_{\alpha \oplus e_{2m}} \\ (-1)^{\theta(\beta, i)} z_{\beta \oplus e_i \oplus e}^* & (-1)^{\phi(\beta)} z_{\beta \oplus e_{2m}} \end{pmatrix},$$

where $\beta = \alpha \oplus e_i \oplus e_{2m} \oplus e$. For $\alpha \oplus \beta = e_i \oplus e_{2m} \oplus e$, calculate $\theta(\alpha, i) + \theta(\beta, i)$ by definition (4). **When i is even**, $\theta(\alpha, i) + \theta(\beta, i) \equiv \text{wt}_{i, 2m}(e_i \oplus e_{2m} \oplus e) + i \equiv 2m - i - 1 + i \equiv 1 \pmod{2}$, which implies $\phi(\alpha) = \phi(\alpha \oplus e_i \oplus e_{2m} \oplus e)$ for even i . **When i is odd**, $\theta(\alpha, i) + \theta(\beta, i) \equiv \text{wt}_{i, 2m}(e_i \oplus e_{2m} \oplus e) + (i-1) + 2 \equiv (2m - i - 1) + (i-1) + 2 \equiv 0 \pmod{2}$, which implies $\phi(\alpha) = \phi(\alpha \oplus e_i \oplus e_{2m} \oplus e) \oplus 1$ for odd i .

Now, we are ready to induce the contradiction. For any $\alpha \in \mathbb{F}_2^n$, let $i = 2l$, $l = 1, 2, \dots, m-1$ and $i = 2l-1$, $l = 1, 2, \dots, m$ separately. We have

$$\begin{aligned} \phi(\alpha) &= \phi\left(\alpha \bigoplus_{l=1}^{m-1} (e_{2l} \oplus e_{2m} \oplus e) \bigoplus_{l=1}^m (e_{2l-1} \oplus e_{2m} \oplus e)\right) \oplus m \\ &= \phi\left(\alpha \oplus e_{2m} \oplus e \bigoplus_{l=1}^{2m-1} e_l\right) \oplus 1 \\ &= \phi(\alpha \oplus e_{2m} \oplus e \oplus e \oplus e_{2m}) \oplus 1 \\ &= \phi(\alpha) \oplus 1, \end{aligned}$$

which is a contradiction! \square

Equipped with the above results, we are able to prove the unexistence of COD with parameter $\left[\binom{2m}{m-1}, 2m, \binom{2m-1}{m-1}\right]$, m odd.

Theorem 3.6. *There does not exist COD with parameter $\left[\binom{2m}{m-1}, 2m, \binom{2m-1}{m-1}\right]$ when m is odd.*

Proof. We prove it by contradiction. Assume there exists a COD \mathcal{O}_z with parameter $\left[\binom{2m}{m-1}, 2m, \binom{2m-1}{m-1}\right]$. Deleting one column, we obtain a COD with parameter $\left[\binom{2m}{m-1}, 2m-1, \binom{2m-1}{m-1}\right]$, which is denoted by \mathcal{O}'_z . By Theorem 3.4,

we know \mathcal{O}'_z can be obtained by equivalence operation over \mathcal{G}_{2m-1} . Since equivalence operation is invertible, apply the inverse operation on \mathcal{O}_z , we obtain a COD $(\mathcal{G}_{2m-1}, \mathcal{L}_{2m})$. By Lemma 3.5, we know it's impossible to add an extra column on \mathcal{G}_{2m-1} still to be orthogonal. \square

The following corollary is a direct consequence of the previous results.

Corollary 3.7. *When $n \equiv 0, 1, 3 \pmod{4}$, CODs with parameter $[p, n, k]$ achieving maximal rate and minimal delay are the same under equivalence operation.*

Proof. When $n \equiv 1, 3 \pmod{4}$, i.e., $n = 2m - 1$ for integer m , COD \mathcal{O}_z with parameter $\left[\binom{2m}{m-1}, 2m - 1, \binom{2m-1}{m-1}\right]$ achieves maximal rate and minimal delay. By Theorem 3.4, we know \mathcal{O}_z is the same as \mathcal{G}_{2m-1} under equivalence operation.

When $n \equiv 0 \pmod{4}$, i.e., $n = 2m$, m even, COD \mathcal{O}_z with parameter $\left[\binom{2m}{m-1}, 2m, \binom{2m-1}{m-1}\right]$ achieves maximal rate and minimal delay. Since by deleting one column of \mathcal{O}_z we obtain a maximal-rate, minimal-delay COD for $n = 2m - 1$, which is equivalent to \mathcal{G}_{2m-1} by Lemma 3.4. By Lemma 3.1, we know the remaining column is uniquely determined regardless of signs.

Following the argument in Lemma 3.5, it's sufficient to prove function $\phi(\alpha)$, $\alpha \in \mathbb{F}_2^{2m}$, $\text{wt}(\alpha) = m + 1$, is uniquely determined up to a negation of all. From the proof of Lemma 3.5, we know $\phi(\alpha) = \phi(\alpha \oplus e_i \oplus e_{2m} \oplus e)$ for even i ; and $\phi(\alpha) = \phi(\alpha \oplus e_i \oplus e_{2m} \oplus e) \oplus 1$ for odd i , where $\alpha(i) = 1$. Again, take integer j such that $(\alpha \oplus e_i \oplus e_{2m} \oplus e)(j) = 1 \Leftrightarrow \alpha(j) = 0$, we can obtain the relationship between $\phi(\alpha)$ and $\phi(\alpha \oplus e_i \oplus e_{2m} \oplus e \oplus e_j \oplus e_{2m} \oplus e) = \phi(\alpha \oplus e_i \oplus e_j)$. Since i, j are taken arbitrarily, we know all relationships between $\phi(\alpha)$ are determined. \square

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References

- [1] S. S. Adams, J. Davis, N. Karst, M. K. Murugan, B. Lee, M. Crawford, C. Greeley, "Novel classes of minimal delay and low PAPR rate 1/2 complex orthogonal designs," *IEEE Trans. Inf. Theory*, vol. 57, no. 4, pp. 2254-2262, Apr. 2011.
- [2] S. S. Adams, N. Karst, M. K. Murugan and T. A. Wysocki, "On transceiver signal linearization and the decoding delay of maximum rate complex orthogonal space-time block codes," *IEEE Trans. Inf. Theory*, vol. 57, no. 6, pp. 3618-3621, Jun. 2011.
- [3] S. S. Adams, N. Karst, and J. Pollack, "The minimum decoding delay of maximum rate complex orthogonal space-time block codes," *IEEE Trans. Inf. Theory*, vol. 53, no. 8, pp. 2677-2684, Aug. 2007.
- [4] S. S. Adams, N. Karst, M. K. Murugan., "The final case of the decoding delay problem for maximum rate complex orthogonal designs," *IEEE Trans. Inf. Theory*, vol. 56, no. 1, pp. 103-122, Jan. 2010.

- [5] S. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. Select. Areas Commun.*, vol. 16, pp. 1451-1458, Oct. 1998.
- [6] A. V. Geramite and N. J. Pullman, "Orthogonal Designs: Quadratic Forms and Hadamard Matrices (Lecture Notes in Pure and Applied Mathematics)," New York: Marcel Dekker, vol. 43, 1979.
- [7] R. L. Graham, D. E. Knuth and O. Patashnik, "Concrete Mathematics: A Foundation for Computer Science," 2nd Edition, Pearson Education, 1994.
- [8] H. Kan and H. Shen, "A counterexample for the open problem on the minimal delays of orthogonal designs with maximal rates," *IEEE Trans. Inf. Theory*, vol. 51, no. 1, pp. 355-359, Jan. 2005.
- [9] X-B. Liang, "Orthogonal designs with maximal rates," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2468-2503, Oct. 2003.
- [10] Y. Li, H. Kan, C. Yuan and H. Ma, "The maximal rates and minimal decoding delay of more general complex orthogonal designs," *Science in China, Series F: Information Sciences* 2010, pp. 1826-1832.
- [11] K. Lu, S. Fu, and X. G. Xia, "Closed-form designs of complex orthogonal space-time block codes of rates $(k+1)/(2k)$ for $2k-1$ or $2k$ transmit antennas," *IEEE Trans. Inf. Theory*, vol. 51, no. 12, pp. 4340-4347, Dec. 2005.
- [12] W. Su, X.-G. Xia, and K. J. R. Lui, "A systematic design of high-rate complex orthogonal space-time block codes," *IEEE Commun. Lett.* vol. 8, no. 6, pp. 380-382, Jun. 2004.
- [13] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inf. Theory*, vol. 45, no. 5, pp. 1456-1467, July 1999.
- [14] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Correction to 'Space-time block codes from orthogonal designs,'" *IEEE Trans. Inf. Theory*, vol. 46, no. 1, Jan. 2000.
- [15] L. C. Tran, T. A. Wysocki, J. Seberry, A. Mertins and S. S. Adams, "Novel constructions of improved square complex orthogonal designs for eight transmit antennas," *IEEE Trans. Inf. Theory*, vol. 55, no. 10, Oct. 2009.
- [16] H. Wang and X-G. Xia, "Upper bounds of rates of complex orthogonal space-time block codes," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2788-2796, Oct. 2003.
- [17] W. Su, X.-G. Xia, "Two generalized complex orthogonal space-time block codes of rate $7/11$ and $3/5$ for 5 and 6 transmit antennas," *IEEE Trans. Inf. Theory*, vol. 49, pp. 313-316, Jan. 2003.